6.5 Properties of Logarithms

Learning Objectives

1. Work with the Properties of Logarithms
2. Write a Logarithmic Expression as a Sum or Difference of Logarithms
3. Write a Logarithmic Expression as a Single Logarithm
4. Evaluate Logarithms Whose Base Is Neither 10 Nor e

Definitions

$\log_a 1 = 0 \longrightarrow a^0 = 1$

$\log_a a = 1 \longrightarrow a^1 = a$

Examples

$\log_{27} \mathbf{1} = 0$

$\log_{53} \mathbf{53} = 1$

Theorem

Properties of Logarithms (Inverse properties)

In the properties given next, *M* and *a* are positive real numbers, $a \neq 1$, and *r* is any real number.

log. base a "cancels" with exp. base a

(1) $a^{\log_a M} = M$

(2) $\log_a a^r = r$

Ex $\log_2(2^{50}) = 50$

Example 1

Using Properties (1) and (2)

(a) $2^{\log_2 \pi} = \pi$

(b) $\log_{0.2} 0.2^{\sqrt{3}} = \sqrt{3}$

(c) $\ln e^{kt} = kt$

Theorem

Properties of Logarithms

In the following properties, *M*, *N*, and *a* are positive real numbers, $a \neq 1$, and *r* is any real number.

The Log of a Product Equals the Sum of the Logs

(3) $$\log_a(MN) = \log_a M + \log_a N$$

$a^n \cdot a^m = a^{n+m}$

The Log of a Quotient Equals the Difference of the Logs

(4) $$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

$\frac{a^n}{a^m} = a^{n-m}$

The Log of a Power Equals the Product of the Power and the Log

(5) $$\log_a\left(M^r\right) = r\log_a M$$

$(a^n)^m = a^{nm}$

Example 2

Writing a Logarithmic Expression as a Sum of Logarithms

Write $\log\left(\overset{M}{x}\sqrt{\overset{N}{x^2+1}}\right), x > 0,$ as a sum of logarithms.

$* \sqrt{x} = x^{\frac{1}{2}}$

Express all powers as factors.

$$\log\left(x\sqrt{x^2+1}\right) = \log x + \log\sqrt{x^2+1}$$

$$= \log x + \log\left(x^2+1\right)^{\frac{1}{2}}$$

$$= \boxed{\log x + \frac{1}{2}\log\left(x^2+1\right)}$$

Example 3

Writing a Logarithmic Expression as a Difference of Logarithms

Write $$\ln\left[\frac{x^2}{(x-1)^3}\right] \quad x > 1$$

as a difference of logarithms. Express all powers as factors.

$$\ln\left[\frac{x^2}{(x-1)^3}\right] = \ln x^2 - \ln(x-1)^3$$

$$= \boxed{2\ln x - 3\ln(x-1)}$$

Example 4

Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a\left[\frac{\sqrt{x^2+1}}{x^3(x+1)^4}\right] \quad x>0$$

as a sum and difference of logarithms. Express all powers as factors.

$$= \log_a\sqrt{x^2+1} - \underbrace{\log_a\left(x^3(x+1)^4\right)}$$

$$= \log_a(x^2+1)^{1/2} - \left[\log_a x^3 + \log_a(x+1)^4\right]$$

$$= \boxed{\frac{1}{2}\log_a(x^2+1) - 3\log_a x - 4\log_a(x+1)}$$

Example 5

Writing Expressions as a Single Logarithm

Write each of the following as a single logarithm.

(a) $\log_a 7 + 4\log_a 3$ $= \log_a 7 + \log_a 3^4 = \log_a(7\cdot 3^4) = \boxed{\log_a 567}$

(b) $\frac{2}{3}\ln 8 - \ln(5^2 - 1)$ $= \ln 8^{2/3} - \ln(5^2-1) = \ln 4 - \ln 24$

$= \ln\left(\frac{4}{24}\right) = \boxed{\ln\left(\frac{1}{6}\right)}$

(c) $\underbrace{\log_a x + \log_a 9 + \log_a(x^2+1)}_{= \log_a(9x(x^2+1))} - \log_a 5$

$= \log_a(9x(x^2+1)) - \log_a 5$

$= \log_a\left(\frac{9x(x^2+1)}{5}\right)$

Theorem

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a}$$

Ex $\log_2 36 = \dfrac{\log 36}{\log 2} = \dfrac{\ln 36}{\ln 2}$

Example 6

Using the Change-of-Base Formula

Approximate: (with your calculator)

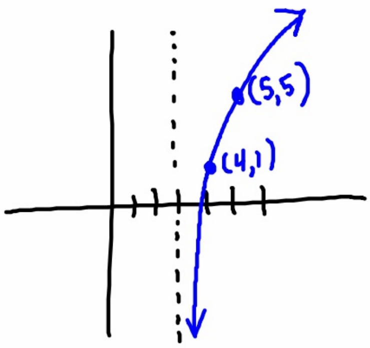
(a) $\log_5 89 \approx 2.7889$

(b) $\log_{\sqrt{2}} \sqrt{5} \approx 2.3219$

Round answers to four decimal places.

Example 7

Use a graphing utility and the change of base theorem to graph the following function.



$$f(x) = 8\log_4(x-3) + 1$$

vert. stretch +8

base 4

horiz. shift +3

vert. shift +1